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On Kähler manifolds with positive orthogonal bisectional curvature

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Abstract

For any irreducible Kähler manifold which admits positive orthogonal bisectional curvature and $C_1 > 0$, if this positivity condition is preserved under the flow, then the underlying manifold is biholomorphic to $\mathbb{CP}^n$.

Keywords: Kähler manifold; Kähler Ricci flow; Positive orthogonal bisectional curvature

Contents

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1. Introduction

The famous Frankel conjecture asserts that any compact Kähler manifold with positive bisectional curvature must be biholomorphic to $\mathbb{CP}^n$. This conjecture was settled affirmatively in early 1980s by two groups of mathematicians independently: Siu–Yau [15] via differential geometry method and Mori [13] by algebraic method. There are many interesting papers following this celebrated work; in particular to understand the classification of Kähler manifolds with non-negative bisectional curvature, readers are referred to N. Mok's work [12] for further references. In 1982, R. Hamilton [10] introduced the Ricci flow as a means to deform any Riemannian metric in a canonical way to an Einstein metric. He particularly showed that, in any 3-dimensional compact manifold, the positive Ricci curvature is preserved by the Ricci flow. Moreover, the Ricci flow deforms the metric more and more toward an Einstein metric. Consequently, he proved that the underlying manifold must be diffeomorphic to S^3 or a finite quotient of S^3. By a theorem of M. Berger in the 1960s, any Kähler Einstein metric with positive bisectional curvature is the Fubini-Study metric (with constant bisectional curvature). A natural and long standing problem for Kähler Ricci flow is: In $\mathbb{CP}^n$, does Kähler Ricci flow converge to the Fubini-Study metric if the initial metric has positive bisectional curvature? There is much interesting work in this direction in 1990s (cf. [1,12]) and the problem was completely settled in 2000 by [7,8] affirmatively. One key idea is the introduction of a series new geometrical functionals which play a crucial role in deriving the bound of scalar curvature, diameter and Sobolev constants etc.

It is well known that the positivity of bisectional curvature is preserved along the Kähler Ricci flow, due to S. Bando [1] in dimension 3 and N. Mok in general dimension [12]. Following the work of N. Mok, in an unpublished work of Cao–Hamilton, they claimed that the positive orthogonal bisectional curvature is also preserved under the Kähler Ricci flow. However, we are NOT assuming this unpublished work of Cao–Hamilton.

In any Kähler manifold, we can split the space of $(1,1)$ forms into two orthogonal components: the line spanned by the Kähler form, and its orthogonal complement $\Lambda_0^{1,1}$. The traceless part of the bisectional curvature can be viewed as an operator acting on this subspace $\Lambda_0^{1,1}$. We say a Kähler metric has 2-positive traceless bisectional curvature if the sum of any two eigenvalues of the traceless bisectional curvature operator in $\Lambda_0^{1,1}$ is positive. In complex dimension 2, this property was studied in [14] where the authors proved that the 2-positive traceless bisectional curvature condition is preserved under the Kähler Ricci flow. This result, as well as the method of proof, is similar to that of H. Chen in Riemannian case. In [6], H. Li and the author prove that the 2-positive traceless bisectional curvature operator is preserved under the Kähler Ricci flow in all dimensions. Moreover, we show that in [6] that Kähler metric with 2-positive traceless bisectional curvature must also have positive orthogonal bisectional curvature. Therefore, if the initial Kähler metric has 2-positive traceless bisectional curvature, then the orthogonal bisectional curvature remains positive on the entire Kähler Ricci flow.

In this paper, we study any Kähler manifold where the positive orthogonal bisectional curvature is preserved on the Kähler Ricci flow. Naturally, we always assume that the first Chern class C_1 is positive. Under this assumption, we first show that various inequalities (convex cone), on the scalar curvature, the Ricci curvature tensor and the holomorphic sectional curvature tensor, are preserved over the Kähler Ricci flow respectively. We then discuss geometric applications of these results. In particular, we prove that any irreducible Kähler manifold with 2-positive traceless bisectional curvature and positive first Chern class must be biholomorphic to $\mathbb{CP}^n$. This can be viewed as a generalization of Siu–Yau [15], Mori's solution [13] of the Frankel conjecture.

Now we state some results on maximum principle first. The proof of the maximum principle results is given in Section 3.

Theorem 1.1. *Along the Kähler Ricci flow, the following statements hold*:

1. *The lower bound of the scalar curvature, if* $\leqslant 0$*, is preserved and improved over time. The lower bound will increase to* 0 *exponentially.*
2. *The upper bound of the scalar curvature grows at most exponentially.*

Theorem 1.2. *If the orthogonal bisectional curvature is positive along the Kähler Ricci flow, then the following hold*:

1. *If the initial metric has positive Ricci curvature, then this condition will be preserved.*
2. *If the initial Ricci curvature is not positive, then the lower bound of the Ricci curvature will increase. As* $t \to \infty$*, this lower bound will at least increase to* 0 *as* $t \to \infty$*.*

Remark 1.3. Theorem 1.1 was proved by R. Hamilton [11] and B. Chow [9] in S^2. Statements in Theorem 1.1 should be known to the experts in the field and the proof in high dimension is similar to 2-d case.

The first part of Theorem 1.2 was observed by Cao and Hamilton first.

Theorem 1.4. *If the positivity of the orthogonal bisectional curvature is preserved along the Kähler Ricci flow, then the lower bound of the holomorphic sectional curvature, if non-positive, is preserved under the flow. Moreover, the lower bound will be improved to* 0 *exponentially over the flow.*

The following theorem is more or less technical.

Theorem 1.5. *On the Kähler manifold where the positivity of the orthogonal bisectional curvature is preserved under the Kähler Ricci flow, suppose that the condition* $\mathrm{Ric} \geqslant \nu > 0$ *is preserved over the entire flow. Then the lower bound of the holomorphic sectional curvature, if* $\leqslant 0$*, will become positive after finite time. Moreover, if the lower bound of bisectional curvature is positive and* $\nu > \frac{1}{2}$*, then this lower bound will be increased over time and approach* $\frac{2\nu-1}{n+1}$ *exponentially.*

The following question is then very natural.

Question/Conjecture 1.6. *Is an irreducible compact Kähler manifold with positive orthogonal bisectional curvature necessarily* $\mathbb{CP}^n$*?*

A related question is: for higher dimension, does a manifold with positive orthogonal bisectional curvature necessarily has first Chern class positive? Apparently, we need to assume $n > 1$. Hopefully, when n is large enough, the positive orthogonal bisectional curvature condition implies the positivity of first Chern class?

Following [8], we have

Theorem 1.7. *On a Kähler Einstein manifold where the positive orthogonal bisectional curvature is preserved over the Kähler Ricci flow, the flow converges to a Kähler Einstein metric exponentially fast.*

Comparing to Theorem 1.1 and Remark 1.9 of [8], we drop the assumption that the initial metric has positive Ricci curvature. Theorem 1.2 implies that, after finite time, the Ricci curvature will be bigger than -1. The energy functional E_1 will be monotonously decreased afterwards. The rest of the proof is the same as in [8]. Here E_1 is one of a set of new geometrical functional introduced in [8].

Invoking a deep theorem of Perelman where he shows that the scalar curvature is uniformly bounded along the Kähler Ricci flow with any smooth initial metric,[1] then we prove

Theorem 1.8. *For any irreducible Kähler manifold which admits positive orthogonal bisectional curvature and $C_1 > 0$, if this positivity condition is preserved under the flow, then the underlying manifold is biholomorphic to $\mathbb{CP}^n$.*

As a corollary, we have

Theorem 1.9. *Any irreducible Kähler manifold with positive 2-traceless bisectional curvature*[2] *and $C_1 > 0$ must be $\mathbb{CP}^n$.*

Following proof of Theorem 1.8 and classification of manifold with non-negative bisectional curvature (cf. [12]), one shall be able to generalize Theorem 1.8 to the case of non-negative orthogonal bisectional curvature positive case.

One unsatisfactory feature of the present proof is that we had to take limit as $t \to \infty$ first in order to show that the bisectional curvature becomes positive after finite time. The argument is a little bit indirect. If a more direct argument can be obtained, then perhaps one can avoid using of Perelman's theorem on the scalar curvature function. The following is a weak, but direct theorem.

Theorem 1.10. *For any Kähler manifold with positive orthogonal bisectional curvature and $C_1 > 0$, if this positivity condition is preserved under the Kähler Ricci flow, the following statements are equivalent*:

1. *There exists a lower bound of the energy functional E_1*;
2. *There exists a lower bound of the Mabuchi energy*;
3. *There is a Kähler Einstein metric in the Kähler class.*

If any of the above holds, then the Kähler Ricci flow converges exponentially fast to the Fubini-Study metric and the underlying manifold is $\mathbb{CP}^n$.

Note that in the proof of this theorem, we do not use Frankel conjecture.

[1] For an updated reference of Perelman's result, one may consult [16].

[2] Positive orthogonal bisectional curvature if we prepare to accept unpublished work of Cao–Hamilton.

2. Basic Kähler geometry

2.1. Setup of notations

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \ldots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} > 0,$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive $(1,1)$-form. Given a Kähler metric ω, its volume form is

$$\omega^n = \frac{1}{n!} (\sqrt{-1}\,)^n \det(g_{i\bar{j}})\, dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

The curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \quad \forall i, j, k, l = 1, 2, \ldots, n.$$

The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature form is

$$\operatorname{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}}(\omega)\, dz^i \wedge dz^{\bar{j}} = -\sqrt{-1}\partial\bar{\partial} \log\det(g_{k\bar{l}}).$$

It is a real, closed $(1,1)$-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda\omega$, for some constant λ. In our setting, we require $\lambda = 1$.

We say that ω is of non-negative bisectional curvature if

$$R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} \geqslant 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of M. We say that ω has a non-negative orthogonal bisectional curvature if

$$R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} \geqslant 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of M such that $(v, w) = (v, Jw) = 0$ where J is the underlying complex structure.

2.2. The Kähler Ricci flow

Now we assume that the first Chern class $c_1(M)$ is positive. The normalized Ricci flow (cf. [2]) on a Kähler manifold X is

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \quad \forall i, j = 1, 2, \ldots, n. \tag{2.1}$$

It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} h_\omega \quad \text{and} \quad \int_X \left(e^{h_\omega} - 1\right)\omega^n = 0.$$

Then the evolution equation for bisectional curvature is

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = {} & \Delta R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\ & - \frac{1}{2}\left(R_{i\bar{p}} R_{p\bar{j}k\bar{l}} + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}}\right). \end{aligned} \tag{2.3}$$

The evolutions equations for Ricci curvature and scalar curvature are

$$\frac{\partial R_{i\bar{j}}}{\partial t} = \Delta R_{i\bar{j}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}} - R_{i\bar{p}} R_{p\bar{j}}, \tag{2.4}$$

$$\frac{\partial R}{\partial t} = \Delta R + R_{i\bar{j}} R_{j\bar{i}} - R. \tag{2.5}$$

One shall note that the Laplacian operator appearing in the above formulas is the Laplace–Beltrami operator on functions. As usual, the flow equation (2.1) or (2.2) is referred as the Kähler Ricci flow on M. It is proved by Cao [2], who followed Yau's celebrated work [17], that the Kähler Ricci flow exists globally for any smooth initial Kähler metric.

3. The maximum principle along the Kähler Ricci flow

In this section, we prove some theorems on the scalar curvature, Ricci curvature and holomorphic sectional curvature via Hamilton's maximum principle on tensors along a geometric flow.

3.1. On the scalar curvature

Here we give a proof to Theorem 1.1.

Proof of Theorem 1.1. Set

$$F(t) = \left|\nabla \frac{\partial \varphi}{\partial t}\right|^2_{\varphi(t)}.$$

Then the evolution equation for $F(t)$ is simple:

$$\frac{\partial}{\partial t} F = \Delta_{\varphi(t)} F + F - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Note that

$$\left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} - R_{\alpha\bar{\beta}}.$$

Recall the evolution equation for the scalar curvature

$$\begin{aligned}\frac{\partial R}{\partial t} &= \Delta R + |\mathrm{Ric} - 1|^2 + 2R - n - R \\ &= \Delta R + \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\beta} + R - n.\end{aligned}$$

Set $h = R - n + F$. Then the evolution equation for h is

$$\frac{\partial h}{\partial t} = \Delta h + h - \left(\frac{\partial\varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial\varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Thus, the upper bound of h grows at most exponentially:

$$h \leqslant \max_{x\in M} h(x,0)e^t.$$

Consequently, we have

$$R(x,t) \leqslant C_1 e^t + n, \quad \forall t > 0.$$

Let $\mu(t)$ be a solution to the ODE

$$\mu'(t) + \mu(t) = 0,$$

with $-\mu(0)$ a lower bound for $R(x,0) > 0$. Then

$$\begin{aligned}\frac{\partial}{\partial t}\big(R + \mu(t)\big) &= \Delta\big(R + \mu(t)\big) + \mathrm{Ric}^2 - \big(R + \mu(t)\big) + \mu(t) + \mu(t)' \\ &\geqslant \Delta\big(R + \mu(t)\big) - \big(R + \mu(t)\big).\end{aligned}$$

Then, the maximal principle now shows that $R + \mu$ is always positive, so in fact $-\mu(t)$ is a lower bound of $R(x,t)$ for all $t \geqslant 0$. In particular, if $\min_{x\in M} R(x,0) < 0$, then

$$R(x,t) \geqslant \min_{x\in M} R(x,0)e^{-t}. \qquad \square$$

3.2. On the Ricci curvature

Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. As before, we set a time dependent function $\mu(t)$ here. We choose $\mu(0)$ such that $R_{i\bar{j}} + \mu g_{i\bar{j}} \geqslant 0$ at time $t = 0$ and let $\mu(t)$ solve the following ODE:

$$\mu'(t) + \mu(\mu + 1) = 0.$$

In other words,

$$\mu(t)=\frac{C}{e^t-C},\quad \text{where } C=\frac{\mu(0)}{\mu(0)+1}.$$

Set

$$\hat{R}_{i\bar{j}}=R_{i\bar{j}}+\mu(t)g_{i\bar{j}}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}\hat{R}_{i\bar{j}}&=\frac{\partial}{\partial t}R_{i\bar{j}}+\mu(t)\frac{\partial}{\partial t}g_{i\bar{j}}+\mu' g_{i\bar{j}}\\&=\Delta_\varphi R_{i\bar{j}}+R_{i\bar{j}k\bar{l}}R_{\bar{k}l}-R_{i\bar{p}}R_{p\bar{j}}+\mu(t)(g_{i\bar{j}}-R_{i\bar{j}})+\mu' g_{i\bar{j}}\\&=\Delta_\varphi \hat{R}_{i\bar{j}}+R_{i\bar{j}k\bar{l}}\hat{R}_{\bar{k}l}-\mu R_{i\bar{j}}-\hat{R}_{i\bar{p}}\hat{R}_{p\bar{j}}+2\mu\hat{R}_{i\bar{j}}-\mu^2 g_{i\bar{j}}+\mu(\mu+1)g_{i\bar{j}}\\&\quad-\mu\hat{R}_{i\bar{j}}+\mu' g_{i\bar{j}}\\&=\Delta_\varphi \hat{R}_{i\bar{j}}+R_{i\bar{j}k\bar{l}}\hat{R}_{\bar{k}l}-\mu\hat{R}_{i\bar{j}}-\hat{R}_{i\bar{p}}\hat{R}_{p\bar{j}}+2\mu\hat{R}_{i\bar{j}}+\mu(\mu+1)g_{i\bar{j}}-\mu\hat{R}_{i\bar{j}}+\mu' g_{i\bar{j}}\\&=\Delta_\varphi \hat{R}_{i\bar{j}}+R_{i\bar{j}k\bar{l}}\hat{R}_{\bar{k}l}-\hat{R}_{i\bar{p}}\hat{R}_{p\bar{j}}+\big(\mu(\mu+1)+\mu'\big)g_{i\bar{j}}.\end{aligned}$$

Then the evolution equation for $\hat{R}_{i\bar{j}}$ is

$$\frac{\partial}{\partial t}\hat{R}_{i\bar{j}}=\Delta_\varphi\hat{R}_{i\bar{j}}+R_{i\bar{j}k\bar{l}}\hat{R}_{k\bar{l}}-\hat{R}_{i\bar{p}}\hat{R}_{p\bar{j}}.$$

Consequently, the non-negativity of $\hat{R}_{i\bar{j}}$ is preserved. This is because at the point where $\hat{R}_{i\bar{j}}$ vanishes at least in one direction, we can show that the right-hand side must be non-negative. In fact, set this direction as $\frac{\partial}{\partial z_1}$ and diagonalize the Ricci curvature at this point. Thus, at this point, we have

$$\hat{R}_{i\bar{j}}=\lambda_j\delta_{ij},\quad i,j=1,2,\ldots,n.$$

Here

$$\lambda_1=0\leqslant\lambda_j,\quad \forall j\geqslant 2.$$

Applying Hamilton's maximum principle for tensors, we have

$$\frac{\partial}{\partial t}\hat{R}_{1\bar{1}}\geqslant R_{1\bar{1}j\bar{j}}\hat{R}_{j\bar{j}}-\hat{R}_{1\bar{1}}\hat{R}_{\bar{1}1}=\sum_{j=2}^{n}R_{1\bar{1}j\bar{j}}\hat{R}_{j\bar{j}}\geqslant 0.$$

Thus, $-\mu(t)<0$ is a lower bound of Ricci curvature (not necessarily optimal) for all time $t\geqslant 0$. $\square$

3.3. On the holomorphic sectional curvature

Suppose $(M, g(0))$ is a Kähler manifold with positive orthogonal bisectional curvature. Roughly speaking, we want to prove that the lower bound of the holomorphic sectional curvature, if not positive, will be preserved or improved under appropriate other conditions. Now we give a proof of Theorem 1.4.

Proof of Theorem 1.4. For any bisectional curvature type tensor $A_{i\bar{j}k\bar{l}}$, we define the operator $\Box$ as

$$\Box A_{i\bar{j}k\bar{l}} = \Delta A_{i\bar{j}k\bar{l}} + A_{i\bar{j}p\bar{q}} A_{q\bar{p}k\bar{l}} - A_{i\bar{p}k\bar{q}} A_{p\bar{j}q\bar{l}} + A_{i\bar{l}p\bar{q}} A_{q\bar{p}k\bar{j}} + A_{i\bar{j}k\bar{l}}$$
$$- \frac{1}{2}(R_{i\bar{p}} A_{p\bar{j}k\bar{l}} + R_{p\bar{j}} A_{i\bar{p}k\bar{l}} + R_{k\bar{p}} A_{i\bar{j}p\bar{l}} + R_{p\bar{l}} A_{i\bar{j}k\bar{p}}).$$

Then the evolution equation for bisectional curvature is

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} &= \Box R_{i\bar{j}k\bar{l}} \\ &= \Delta R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\ &\quad - \frac{1}{2}(R_{i\bar{p}} R_{p\bar{j}k\bar{l}} + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}}).\end{aligned}$$

Set

$$S_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \mu(t)(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{j\bar{k}}) = R_{i\bar{j}k\bar{l}} - \mu(g * g)_{i\bar{j}k\bar{l}}$$

and

$$S_{k\bar{l}} = R_{k\bar{l}} - \mu(n+1) g_{k\bar{l}}.$$

At the point where $g_{i\bar{j}} = \delta_{i\bar{j}}$, we can re-write the evolution equation for R as

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} &= \Delta\big(S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}}\big) + \big(S_{i\bar{j}p\bar{q}} + \mu(g*g)_{i\bar{j}p\bar{q}}\big)\big(S_{q\bar{p}k\bar{l}} + \mu(g*g)_{q\bar{p}k\bar{l}}\big) \\ &\quad - \big(S_{i\bar{p}k\bar{q}} + \mu(g*g)_{i\bar{p}k\bar{q}}\big)\big(S_{p\bar{j}q\bar{l}} + \mu(g*g)_{p\bar{j}q\bar{l}}\big) + S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}} \\ &\quad + \big(S_{i\bar{l}p\bar{q}} + \mu(g*g)_{i\bar{l}p\bar{q}}\big)\big(S_{q\bar{p}k\bar{j}} + \mu(g*g)_{q\bar{p}k\bar{j}}\big) \\ &\quad - \frac{1}{2}(R_{i\bar{p}} S_{p\bar{j}k\bar{l}} + R_{p\bar{j}} S_{i\bar{p}k\bar{l}} + R_{k\bar{p}} S_{i\bar{j}p\bar{l}} + R_{p\bar{l}} S_{i\bar{j}k\bar{p}}) \\ &\quad - \frac{\mu}{2}\big(R_{i\bar{p}}(g*g)_{p\bar{j}k\bar{l}} + R_{p\bar{j}}(g*g)_{i\bar{p}k\bar{l}} + R_{k\bar{p}}(g*g)_{i\bar{j}p\bar{l}} + R_{p\bar{l}}(g*g)_{i\bar{j}k\bar{p}}\big) \\ &= \Box S_{i\bar{j}k\bar{l}} + 2\mu S_{i\bar{j}k\bar{l}} + \mu(S_{i\bar{j}} g_{k\bar{l}} + S_{k\bar{l}} g_{i\bar{j}}) + \mu^2 (g*g)_{i\bar{j}p\bar{q}} (g*g)_{q\bar{p}k\bar{l}} \\ &\quad - 2\mu S_{i\bar{j}k\bar{l}} - \mu(S_{i\bar{l}k\bar{j}} + S_{i\bar{l}k\bar{j}}) - \mu^2 (g*g)_{i\bar{p}k\bar{q}} (g*g)_{p\bar{j}q\bar{l}}\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mu S_{i\bar{l}k\bar{j}} + \mu(S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}}) + \mu^2(g*g)_{i\bar{l}p\bar{q}}(g*g)_{q\bar{p}k\bar{j}} \\
&\quad + \mu(g*g)_{i\bar{j}k\bar{l}} - \mu(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu^2\big((n+2)g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - 2g_{i\bar{j}}g_{k\bar{l}} - 2g_{i\bar{l}}g_{k\bar{j}} + (n+2)g_{i\bar{l}}g_{k\bar{j}} + g_{i\bar{j}}g_{k\bar{l}}\big) \\
&\quad + \mu(g*g)_{i\bar{j}k\bar{l}} \\
&\quad + \mu\big((S_{i\bar{j}} - R_{i\bar{j}})g_{k\bar{l}} + (S_{k\bar{l}} - R_{k\bar{l}})g_{i\bar{j}} + (S_{i\bar{l}} - R_{i\bar{l}})g_{k\bar{j}} + (S_{k\bar{j}} - R_{k\bar{j}})g_{i\bar{l}}\big) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\big((n+1)\mu + 1\big)(g*g)_{i\bar{j}k\bar{l}} \\
&\quad + \mu\big((S_{i\bar{j}} - R_{i\bar{j}})g_{k\bar{l}} + (S_{k\bar{l}} - R_{k\bar{l}})g_{i\bar{j}} + (S_{i\bar{l}} - R_{i\bar{l}})g_{k\bar{j}} + (S_{k\bar{j}} - R_{k\bar{j}})g_{i\bar{l}}\big) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\big((n+1)\mu + 1\big)(g*g)_{i\bar{j}k\bar{l}} \\
&\quad - \mu(n+1)\mu(g_{i\bar{j}}g_{k\bar{l}} + g_{k\bar{l}}g_{i\bar{j}} + g_{i\bar{l}}g_{k\bar{j}} + g_{k\bar{j}}g_{i\bar{l}}) \\
&= \Box S_{i\bar{j}k\bar{l}} + \mu\big(1 - (n+1)\mu\big)(g*g)_{i\bar{j}k\bar{l}}.
\end{aligned}
$$

Notice that $R_{i\bar{j}k\bar{l}} = S_{i\bar{j}k\bar{l}} + \mu(g*g)_{i\bar{j}k\bar{l}}$ and

$$
\begin{aligned}
&\frac{\partial}{\partial t}\big(\mu(g*g)_{i\bar{j}k\bar{l}}\big) \\
&\quad = \mu'(g*g)_{i\bar{j}k\bar{l}} + \mu\big(g_{i\bar{j}}(g_{k\bar{l}} - R_{k\bar{l}}) + g_{i\bar{l}}(g_{k\bar{j}} - R_{k\bar{j}}) + g_{k\bar{l}}(g_{i\bar{j}} - R_{i\bar{j}}) + g_{k\bar{j}}(g_{i\bar{l}} - R_{i\bar{l}})\big) \\
&\quad = \mu'(g*g)_{i\bar{j}k\bar{l}} + 2\mu\big(1 - (n+1)\mu\big)(g*g)_{i\bar{j}k\bar{l}} - \mu(S_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{j}}S_{k\bar{l}} + S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}}),
\end{aligned}
$$

we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Box\right)S_{i\bar{j}k\bar{l}} &= \big(\mu\big((n+1)\mu - 1\big) - \mu'\big)(g*g)_{i\bar{j}k\bar{l}} + \mu(S_{i\bar{j}}g_{k\bar{l}} + S_{k\bar{l}}g_{i\bar{j}} + S_{i\bar{l}}g_{k\bar{j}} + S_{k\bar{j}}g_{i\bar{l}}) \\
&= \big(\mu\big((n+1)\mu - 1\big) - \mu'\big)(g*g)_{i\bar{j}k\bar{l}} + \mu(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}) \\
&\quad - 2\mu^2(n+1)(g*g)_{i\bar{j}k\bar{l}} \\
&= \big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big)(g*g)_{i\bar{j}k\bar{l}} \\
&\quad + \mu(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}).
\end{aligned}
$$

Note that we assume that the positivity of the orthogonal bisectional curvature is preserved by the Kähler Ricci flow. Suppose $\mu(t) < 0$ is the lower bound of the evolved bisectional curvature such that $S_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \mu(t)(g*g)_{i\bar{j}k\bar{l}} \geqslant 0$. At the point and time where $S_{i\bar{j}k\bar{l}}$ reaches 0, this minimum is realized by a holomorphic sectional curvature. Without loss of generality, we may assume that it is $S_{1\bar{1}1\bar{1}}(p, t_0) = 0$. Then at $t = t_0$,

$$R_{1\bar{1}1\bar{1}} = 2\mu.$$

We can diagonalize so that

$$R_{1\bar{1}k\bar{l}}(p, t) = \lambda_k \delta_{kl}, \quad \forall k, l = 1, 2, \ldots, n.$$

Here $\lambda_1 = 2\mu$ and

$$A = R_{1\bar{1}}(p,t) = \sum_{k=1}^{n} \lambda_k = +2\mu + \sum_{k=2}^{n} \lambda_k \geqslant c.$$

As in the Mok's argument [12], note that

$$\begin{aligned}\Box S_{1\bar{1}1\bar{1}}\big|_{(p,t)} &\geqslant 2S_{1\bar{1}p\bar{q}}S_{q\bar{p}1\bar{1}} - S_{1\bar{p}1\bar{q}}S_{p\bar{1}q\bar{1}}\\ &\geqslant S_{1\bar{1}p\bar{q}}S_{q\bar{p}1\bar{1}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{\partial}{\partial t}S_{1\bar{1}1\bar{1}}\bigg|_{(p,t)} &\geqslant S_{1\bar{1}k\bar{l}}S_{\bar{k}l\bar{1}1} + \big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big)(g*g)_{1\bar{1}1\bar{1}} + 2A\mu(g*g)_{1\bar{1}1\bar{1}}\\ &= \sum_{k=2}^{n}(\lambda_k - \mu)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &\geqslant \frac{1}{n-1}\left(\sum_{k=2}^{n}\lambda_k - (n-1)\mu\right)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &= \frac{1}{n-1}\big(A - (n+1)\mu\big)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &= \frac{A^2}{n-1} - 2\mu\left(1 - \frac{n-3}{n-1}A\right) + \frac{n+1}{n-1}(3-n)\mu^2 - 2\mu'.\end{aligned}$$

From the last expression, it shows that for $n = 2, 3,$ the lower bound of holomorphic sectional curvature is preserved (when the absolute lower bound is big enough). However, more should be true. In fact, when the minimum of $S_{i\bar{j}k\bar{l}}$ is achieved by holomorphic sectional curvature, by an argument of linear algebra (cf. Lemma 3.1 and inequality (3.2) below; this appears explicitly in [3] (which eventually is due to R. Hamilton)), we can squeeze a little more to obtain

$$\begin{aligned}\frac{\partial}{\partial t}S_{1\bar{1}1\bar{1}}\bigg|_{(p,t)} &\geqslant 2S_{1\bar{1}k\bar{l}}S_{\bar{k}l\bar{1}1} + \big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big)(g*g)_{1\bar{1}1\bar{1}} + 2A\mu(g*g)_{1\bar{1}1\bar{1}}\\ &= 2\sum_{k=2}^{n}(\lambda_k - \mu)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &\geqslant \frac{2}{n-1}\left(\sum_{k=2}^{n}\lambda_k - (n-1)\mu\right)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &= \frac{2}{n-1}\big(A - (n+1)\mu\big)^2 + 2\big(\mu\big(-(n+1)\mu - 1\big) - \mu'\big) + 4A\mu\\ &= \frac{2A^2}{n-1} - 2\mu\left(1 + \frac{4}{n-1}A\right) + \frac{4(n+1)}{n-1}\mu^2 - 2\mu'\end{aligned}$$

$$= \frac{2}{n-1}(A-2\mu)^2 - \frac{8\mu^2}{n-1} - 2\mu + \frac{4(n+1)}{n-1}\mu^2 - 2\mu'$$
$$= \frac{2}{n-1}(A-2\mu)^2 + 4\mu^2 - 2\mu - 2\mu'.$$

Consequently, the lower bound of holomorphic sectional curvature is preserved and gradually improves to 0 as $t \to \infty$. No assumption on Ricci curvature needed to prove the preservation of lower bound here. However, if a positive lower bound of Ricci curvature is preserved, then the holomorphic sectional curvature shall become positive after finite time. □

Lemma 3.1. *Let A, M, N and C be $n \times n$ complex matrices, and A, C, N Hermitian symmetric and M symmetric. Suppose for any $x, y \in \mathbb{C}^n$,*

$$Q(x, y) = A_{i\bar{j}} x^i \bar{x}^j + 2\,\mathrm{Re}\big(M_{ij} x^i y^j + N_{i\bar{j}} x^i \bar{y}^j\big) + C_{i\bar{j}} y^i \bar{y}^j \geqslant 0.$$

Then we have

$$A_{i\bar{j}} C_{j\bar{i}} \geqslant M_{ij}\overline{M_{ij}} + N_{i\bar{j}}\overline{N_{i\bar{j}}}.$$

For the convenience of readers, we will give a short proof of this lemma in the end of this section.

Now we indicate how we can apply this lemma to the preceding proof. Suppose that $S_{i\bar{j}k\bar{l}}$ achieve minimum in the direction $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_{\bar{1}}}$. Then, for any

$$x = \sum_{i=1}^n x^i \frac{\partial}{\partial z_i}, \qquad y = \sum_{i=1}^n y^i \frac{\partial}{\partial z_i},$$

we have

$$\frac{\partial^2}{\partial \epsilon^2} S\left(\frac{\partial}{\partial z_1} + \epsilon x, \frac{\partial}{\partial z_{\bar{1}}} + \epsilon \bar{x}, \frac{\partial}{\partial z_1} + \epsilon y, \frac{\partial}{\partial z_{\bar{1}}} + \epsilon \bar{y}\right)\bigg|_{\epsilon=0} \geqslant 0.$$

In other words, we have

$$S_{1\bar{1}k\bar{l}} y^k y^{\bar{l}} + S_{i\bar{j}1\bar{1}} x^i x^{\bar{j}} + 2\,\mathrm{Re}\big(S_{1\bar{k}1\bar{j}} x^{\bar{k}} y^{\bar{j}} + S_{1\bar{k}j\bar{1}} x^{\bar{k}} y^j\big) \geqslant 0.$$

Applying the lemma above, we have

$$S_{1\bar{1}k\bar{l}} S_{\bar{k}l\bar{1}1} \geqslant S_{1\bar{k}1\bar{j}} S_{\bar{1}k\bar{1}j} + S_{1\bar{k}j\bar{1}} S_{\bar{1}k\bar{j}1} \tag{3.1}$$
$$\geqslant S_{1\bar{k}1\bar{j}} S_{\bar{1}k\bar{1}j} + S_{1\bar{1}j\bar{k}} S_{\bar{1}1\bar{j}k}. \tag{3.2}$$

Now we give a proof to Theorem 1.5.

Proof of Theorem 1.5. First we assume that holomorphic sectional curvature is still negative somewhere in M and

$$R_{i\bar{j}} \geqslant \nu g_{i\bar{j}} > 0.$$

Following the notation of the previous proof in this section, we have (at the minimum of the holomorphic sectional curvature)

$$\begin{aligned}\frac{\partial}{\partial t} S_{1\bar{1}1\bar{1}}\Big|_{(p,t)} &\geqslant \frac{2}{n-1}(A-2\mu)^2+4\mu^2-2\mu-2\mu' \\ &\geqslant \frac{2}{n-1}\nu^2-2\mu'.\end{aligned}$$

The last inequality holds since $\mu<0$. This shows that in finite time, the holomorphic sectional curvature will become positive everywhere.

Now we return to the case where the bisectional curvature is already positive. By assumption, we have

$$R_{i\bar{j}} \geqslant \nu g_{i\bar{j}}, \quad \frac{1}{2}<\nu<1,$$

and the bisectional curvature

$$R_{i\bar{j}k\bar{l}} \geqslant \mu(t)(g*g)_{i\bar{j}k\bar{l}}, \quad \text{where } \mu(0)\geqslant 0.$$

Then,

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\mu(g*g)_{i\bar{j}k\bar{l}}\big) \\ &\quad = \mu'(g*g)_{i\bar{j}k\bar{l}}+\mu\big(g_{i\bar{j}}(g_{k\bar{l}}-R_{k\bar{l}})+g_{i\bar{l}}(g_{k\bar{j}}-R_{k\bar{j}})+g_{k\bar{l}}(g_{i\bar{j}}-R_{i\bar{j}})+g_{k\bar{j}}(g_{i\bar{l}}-R_{i\bar{l}})\big) \\ &\quad \leqslant \mu'(g*g)_{i\bar{j}k\bar{l}}+2\mu(1-\upsilon)(g*g)_{i\bar{j}k\bar{l}}.\end{aligned}$$

Consequently, we have

$$\left(\frac{\partial}{\partial t}-\Box\right)S_{i\bar{j}k\bar{l}} \geqslant \big(\mu\big(2\upsilon-(n+1)\mu-1\big)-\mu'\big)(g*g)_{i\bar{j}k\bar{l}}.$$

Let

$$\mu(t)=\frac{Ce^{at}}{Ce^{at}+1}\cdot a, \quad \text{where } C=\frac{\mu(0)}{1-\mu(0)} \text{ and } a=\frac{2\nu-1}{n+1}.$$

Then

$$\left(\frac{\partial}{\partial t}-\Box\right)S_{i\bar{j}k\bar{l}} \geqslant 0. \qquad \Box$$

Now we give a concise proof of Lemma 3.1.

Proof of Lemma 3.1. Let X^1 and X^2 be the real and imaginary parts of a matrix X, respectively, and

$$E=\begin{pmatrix} A^1 & A^2 \\ -A^2 & A^1\end{pmatrix}, \qquad G=\begin{pmatrix} C^1 & C^2 \\ -C^2 & C^1\end{pmatrix},$$

$$F = \begin{pmatrix} M^1 + N^1 & -M^2 + N^2 \\ -M^2 - N^2 & -M^1 + N^1 \end{pmatrix}.$$

Since M is symmetric and N is Hermitian, F is symmetric. The real matrix corresponding to the form $Q(x, y)$ is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

By Lemma 3.2 below, we have

$$\operatorname{tr}(EG) \geqslant \operatorname{tr}(F^2) = \operatorname{tr}(FF^T),$$

i.e.

$$A_{i\bar{j}} C_{j\bar{i}} \geqslant M_{ij} \overline{M_{ij}} + N_{ij} \overline{N_{ij}}. \qquad \square$$

Lemma 3.2. *Let A, B and C be three $n \times n$ real matrices, and A, C are symmetric. If*

$$G = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \geqslant 0,$$

then

$$\sum_{i,j} A_{ij} C_{ji} \geqslant \sum_{i,j} |B_{ij}|^2.$$

Proof. Let

$$H = \begin{pmatrix} C & -B \\ -B^T & A \end{pmatrix},$$

then $H \geqslant 0$. In fact, since $G \geqslant 0$, for any $x, y \in \mathbb{R}^n$,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} x^T \\ y^T \end{pmatrix} \geqslant 0.$$

By direct calculation,

$$xAx^T + 2xBy^T + yCy^T \geqslant 0. \tag{3.3}$$

This is to say, for any $x, y \in \mathbb{R}^n$,

$$xAx^T - 2xBy^T + yCy^T \geqslant 0,$$

so

$$\begin{pmatrix} y & x \end{pmatrix} \begin{pmatrix} C & -B \\ -B^T & A \end{pmatrix} \begin{pmatrix} y^T \\ x^T \end{pmatrix} \geqslant 0.$$

Hence $H \geqslant 0$, and

$$\operatorname{tr}(GH) \geqslant 0.$$

Note that

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} C & -B \\ -B^T & A \end{pmatrix} = \begin{pmatrix} AC - BB^T & * \\ * & CA - B^T B \end{pmatrix}.$$

Then

$$\operatorname{tr}(AC) \geqslant \operatorname{tr}(BB^T). \qquad \square$$

4. Proof of Theorem 1.8

By Theorem 1.2, Ricci curvature is uniformly bounded from below over the entire flow. According to Perelman, the scalar curvature is uniformly bounded over the entire flow. Consequently, the Ricci curvature is uniformly bounded from below and above. Since orthogonal bisectional curvature is positive and the holomorphic sectional curvature bounded from below (Theorem 1.4), then whole bisectional curvature is uniformly bounded over the entire Kähler Ricci flow. By Perelman's no local collapsing lemma when curvature is bounded, we conclude that the diameter is uniformly bounded from above. Consequently, the injectivity radius must have a uniform positive lower bound over the flow. Suppose that J is the underlying complex structure of M. For any time sequence $\{t_i,\ i \in \mathbb{N}\}$, there exist a subsequence $\{g(t_i),\ i \in \mathbb{N}\}$ and a sequence of diffeomorphisms $\rho_i : M \to M$ such that the sequence of pointed manifolds $\{(M, \rho_i^* g(t_i), J_i = \rho_i^* J),\ i \in \mathbb{N}\}$ converges to a Kähler Ricci soliton g_∞ with non-negative bisectional curvature in M, perhaps with a different complex structure J_∞. Using Theorem 1.4, the bisectional curvature of the limit metric g_∞ must be non-negative. Since g_∞ is a Kähler Ricci soliton, then either the Ricci curvature of g_∞ has no null direction at all or the Kähler manifold will be split into a product of at least two Kähler sub-manifolds. Since we assume that M is irreducible, this implies that the Ricci curvature $R_{i\bar{j}}(g_\infty) > 0$ for any Kähler Ricci soliton arising this way. Since M is compact, then there exists a small constant $\epsilon_0 > 0$ such that $R_{i\bar{j}}(g_\infty) > \epsilon_0$. It is then straightforward to see that there is a positive constant $\epsilon_0 > 0$ which does not depend on which time subsequences we choose. In other words, for $t \geqslant t_0$ big enough, the Ricci curvature is already positive; moreover, a positive uniform lower bound is a priori preserved over the Kähler Ricci flow after $t > t_0$. Following Theorem 1.5, the holomorphic sectional curvature must become positive after finite time beyond $t = t_0$. Since the orthogonal bisectional curvature is always positive before $t = \infty$, we show that the evolved Kähler metric $g(t)$, after at most finite time, must have positive bisectional curvature. Appealing to the Frankel conjecture, M is $\mathbb{CP}^n$. Then, the exponential convergence of the flow follows our earlier work [7,8].

5. Proof of Theorem 1.10

First, let us state a parabolic version of Moser iteration lemma (cf. Lemma 4.7 in [8], also [5]) without proof.

Lemma 5.1. *If the Poincaré constant and the Sobolev constant of the evolving Kähler metrics $g(t)$ are both uniformly, and if a non-negative function u satisfying the following inequality*

$$\frac{\partial}{\partial t}u \leqslant \Delta u + f(x,t)u, \quad \forall a < t < b,$$

where $|f|_{L^p(M,g(t))}$ ($p > \frac{n+2}{2}$) is uniformly bounded by some constant c, then for any $\lambda \in (0,1)$ fixed, we have

$$\max_{(1-\lambda)a+\lambda b \leqslant t \leqslant b} u \leqslant C(c, b-a, \lambda) \int_a^b \int_M u.$$

Now we prove Theorem 1.10.

Proof of Theorem 1.10. We want to prove that each of the three conditions implies the flow converges to a Fubini-Study metrics exponentially. Naturally, this breaks into three cases.

Case 1. Let us first suppose that E_1 has a lower bound. Theorem 1.2 implies that $\mathrm{Ric}(g(t)) > -1$ holds after finite time. From then on, the energy functional E_1 is monotonic decreasing afterwards. Since E_1 is assumed to have a lower bound, the quantity

$$\begin{aligned}\epsilon(t) = E_1(\infty) - E_1(t) &= \int_t^\infty \frac{dE_1}{dt}\,dt \\ &= \int_{t_0}^\infty dt \int_M \left|\mathrm{Ric}\big(g(t)\big) - \omega\big(g(t)\big)\right|^2 \omega\big(g(t)\big)^n \leqslant \epsilon(n)\end{aligned}$$

tends to zero as $t \to \infty$. According to Theorem 1.4 and the fact that the scalar curvature is uniformly bounded, the Riemannian curvature is uniformly bounded from below and above on the entire Kähler Ricci flow. Therefore, the diameter is uniformly bounded on the Kähler Ricci flow as in [8]. Consequently, the injectivity radius must have a uniformly positive lower bound over the entire flow. In other words, both Sobolev constant and Poincaré constant of the evolving metrics are uniformly bounded from above. According to the iteration Lemma 5.1, the Ricci curvature is uniformly pinched toward identity by the constant $\epsilon(t) \to 0$ such that we have

$$1 - \epsilon(t) < \mathrm{Ric}\big(g(t)\big) \leqslant 1 + \epsilon(t), \quad \forall t > t_0.$$

In particular, let $\nu(t)$ to be the lower bound of $\mathrm{Ric}(g(t))$. Then $\lim_{t\to\infty} \nu(t) = 1$. Appealing to Theorem 1.5, the lower bound of the bisectional curvature eventually improves to $\frac{1}{n+1}$. However, at any point $p \in M$, adopting an orthonormal frame, we have

$$\begin{aligned}R_{i\bar{i}}(p) &= \sum_{k=1}^n R_{i\bar{i}k\bar{k}} \\ &\geqslant \sum_{1\leqslant k\leqslant n;\, k\neq i} R_{i\bar{i}k\bar{k}} + R_{i\bar{i}i\bar{i}}.\end{aligned}$$

Thus, if the lower bound of the bisectional curvature improves to $\frac{1}{n+1}$ and the Ricci tensor approaches to the identity, then the full bisectional curvature approaches to the bisectional curvature of a Fubini-Study metric. Consequently, the underlying manifold is $\mathbb{CP}^n$. Following Theorem [8], the flow converges exponentially fast to a metric with constant bisectional curvature.

Case 2. Let us assume now that the Mabuchi energy has a uniform lower bound. Then, we have

$$\int_0^\infty dt \int_M \left|\nabla \frac{\partial \varphi}{\partial t}\right|^2_{\varphi(t)} \omega^n_{\varphi(t)} \leqslant C.$$

The evolution equation for $F(t) = |\nabla \frac{\partial \varphi}{\partial t}|^2_{\varphi(t)}$ is simple:

$$\frac{\partial}{\partial t} F = \Delta_{\varphi(t)} F + F - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} - \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}}.$$

Following the iteration Lemma 5.1, we have

$$F = \left|\nabla \frac{\partial \varphi(t)}{\partial t}\right|^2_{\varphi(t)} \to 0.$$

Consider the evolution of $\int_M F$ over time.

$$\begin{aligned} \frac{d}{dt} \int_M F \omega^n_{\varphi(t)} = \int_M F \omega^n_{\varphi(t)} + \int_M F(n-R) \omega^n_{\varphi(t)} - \int_M \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} \omega^n_{\varphi(t)} \\ - \int_M \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\beta} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\bar{\beta}} \omega^n_{\varphi(t)}. \end{aligned}$$

Since the scalar curvature is uniformly bounded and $F \to 0$ pointwisely, we have

$$\begin{aligned} &\lim_{A\to\infty} \int_A^{A+1} dt \int_M \left|\mathrm{Ric}\big(g(t)\big) - \omega\big(g(t)\big)\right|^2 \omega\big(g(t)\big)^n \\ &= \lim_{A\to\infty} \int_A^{A+1} dt \int_M \left(\frac{\partial \varphi}{\partial t}\right)_{\alpha\bar{\beta}} \cdot \left(\frac{\partial \varphi}{\partial t}\right)_{\bar{\alpha}\beta} \omega^n_{\varphi(t)} \\ &= 0. \end{aligned}$$

Following the iteration lemma again, we obtain

$$1 - \epsilon(t) < \mathrm{Ric}\big(g(t)\big) \leqslant 1 + \epsilon(t), \quad \forall t > t_0.$$

Appealing to Theorem 1.5, the bisectional curvature eventually pinches toward that of Fubini-Study metric. Therefore the underlying manifold is $\mathbb{CP}^n$. Following Theorem [8], the flow converges exponentially fast to a metric with constant bisectional curvature.

Case 3. Let us assume that there exists a Kähler Einstein metric in the canonical Kähler class. By a well-known theorem of Bando–Mabuchi [1], the Mabuchi energy has a uniform lower bound in this class. We can reduce this to Case 2 before and prove the exponential convergence of the Kähler Ricci flow.

We then complete our proof in all three cases. □

6. Future problems

The following questions are interesting:

1. Theorem 1.8 is an extension of Frankel conjecture. Does there exists a direct proof as in Siu–Yau, Mori's well-known theorem?
2. In this paper, we enumerate quite a few new maximum principle theorem. Is the lower bound of Ricci curvature is always preserved in Kähler Ricci flow? Perleman's work seems to suggest that this is true.
3. It is known that the scalar curvature is bounded from above by exponential function over the Kähler Ricci flow. Does the same hold for bisectional curvature?
4. How can one derive the bound of the scalar curvature without using Perelman's deep result?
5. In this paper, we use Perelman's result to derive scalar curvature bound. We obtain a positive lower bound control on Ricci curvature after finite time via an argument of taking limit. Is it possible to have a more direct argument?
6. Does a compact Kähler manifold with positive orthogonal bisectional curvature necessarily have positive first Chern class (except in dimension 1)? The answer should be yes, at least when dimension is high enough.
7. What happens in the case of positive 2 curvature operator in the sense of H. Chen [4]?
8. Is there a (negative) lower bound of the holomorphic sectional curvature which is preserved under the Kähler Ricci flow? This lower bound will depend on the initial Kähler metric. My inclination is that the answer shall be affirmative.

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